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Developmental foundations of children's fraction magnitude knowledge

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ABSTRACT

The conceptual insight that fractions represent magnitudes is a critical yet daunting step in children's mathematical development, and knowledge of fraction magnitudes influences children's later mathematics learning including algebra. In this study, longitudinal data were analyzed to identify the mathematical knowledge and domain-general competencies that predicted 8th and 9th graders' ($n = 122$) knowledge of fraction magnitudes and its cross-grade gains. Performance on the fraction magnitude measures predicted 9th grade algebra achievement. Understanding and fluently identifying the numerator-denominator relation in 7th grade emerged as the key predictor of later fraction magnitudes knowledge in both 8th and 9th grades. Competence at using fraction procedures, knowledge of whole number magnitudes, and the central executive contributed to 9th but not 8th graders' fraction magnitude knowledge, and knowledge of whole number magnitude contributed to cross-grade gains. The key results suggest fluent processing of numerator-denominator relations presages students' understanding of fractions as magnitudes and that the integration of whole number and fraction magnitudes occurs gradually.

1. Introduction

Children's understanding of fractions is central to their mathematical development (National Mathematics Advisory Panel, 2008; NMAP). It is a critical step in extending their understanding of numbers from whole numbers to other rational numbers, and is foundational to advanced mathematics learning including algebra, calculus, and statistics (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; NMAP, 2008; Siegler et al., 2012). Moreover, competence with arithmetic in general, which includes fractions, contributes to employment and wage opportunities in adulthood (Bynner, 1997). Unfortunately, fractions are one of the most challenging mathematical concepts for children to learn (Hoffer, Venkataraman, Hedberg, & Shagle, 2007; Ni & Zhou, 2005; NMAP, 2008). Students in the U.S. typically begin to learn fractions before 4th grade, but about half of 8th graders still cannot order the quantities represented by three fractions, and many of them cannot correctly answer that the sum of 12/13 and 7/8 is closer to 2 than to 19 or 21 (see also Bailey, Siegler, & Geary, 2014; Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Hecht and Vagi, 2010; Mazzocco & Devlin, 2008; Siegler and Pyke, 2013; Siegler, Thompson, & Schneider, 2011). A nationally-representative survey of high school algebra teachers indicated that students'

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poor understanding of fractions and rational numbers more generally was a critical impediment to learning algebra (NMAP, 2008).

A more complete understanding of the prior knowledge and cognitive factors that contribute to fractions learning has the potential to inform instructional and remedial approaches and through this improve the long-term prospects of struggling students (Ni & Zhou, 2005; Siegler et al., 2011; Vosniadou, 2014). Many previous studies have focused on young children's natural number bias, that is, the ways in which their conceptual understanding of whole numbers (e.g., each number has a successor) interferes with their learning of rational numbers (e.g., Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). Children's emerging competence with rational numbers requires an inhibition of conflicting whole number concepts (e.g., multiplication of two numbers results in a larger number) (Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015), and the insight that whole numbers and rational numbers also share a common, conceptual feature – they represent numerical magnitudes (Siegler et al., 2011). Our focus is on this latter aspect of competence with rational numbers, fractions in particular, and on the grades in which this and any associated deficits will directly undermine the learning of algebra (NMAP, 2008). Using data from a comprehensive longitudinal study of children's mathematical development, we show that 8th and 9th graders' understanding of how fractions are situated on the number line predicts 9th grade algebra achievement and identify the foundational mathematical knowledge and domain-general competencies that contribute to their competence with the fractions number line.

1.1. Foundational knowledge

Children's fractions competencies can be categorized into conceptual and procedural knowledge (Bailey et al., 2014, 2015; Byrnes & Wasik, 1991; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). Conceptual knowledge of fractions refers to the understanding of the nature and mathematical properties of fractions, including that a fraction consists of a numerator and a denominator, its magnitude is determined by the numerator-denominator relation, and that these magnitudes can be ordered on a number line (Hecht & Vagi, 2010; Jordan et al., 2013; Siegler, Fazio, Bailey, & Zhou, 2013; Siegler et al., 2011; Vamvakoussi & Vosniadou, 2004). Procedural knowledge, in contrast, refers to understanding of the arithmetic algorithms that can be applied to fractions and their accurate use during problem solving (Bailey et al., 2015; Byrnes & Wasik, 1991). Both conceptual and procedural knowledge are essential to competence with fractions, and their development may be mutually reinforcing (Bailey et al., 2015; Rittle-Johnson et al., 2001; Siegler et al., 2011), although some children become competent with procedures before they conceptually understand fractions and vice versa (Hallett, Nunes, & Bryant, 2010).

As noted, with a recent integrated model of numerical development, Siegler and colleagues state that children's central conceptual insight occurs when they understand that fractions, as with whole numbers, represent magnitudes (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014). Identifying the knowledge foundational to this insight will facilitate the development of efficient approaches for fractions instruction. This knowledge likely includes children's understanding of the relation between the numerator and denominator. A poor understanding of this relation is common and reflected in children's treatment of the numerator and the denominator as independent whole numbers, one aspect of the whole number bias (Gelman & Williams, 1998; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004, 2010). Notably, although the understanding of the numerator-denominator relation and the understanding of fractions magnitudes are conceptually related, they are not identical. The former precedes and does not necessarily involve the latter. Taking fraction comparison as an example, one can judge 1/3 is numerically larger than 1/4 without necessarily representing the magnitude of each of the two fractions or being able to situate their relative position on the number line, using a simple heuristic: "when the numerator is given, the fraction value negatively varies with the denominator". Procedural knowledge of fractions may also be foundational for understanding fractions magnitudes because correct use of fraction arithmetic provides information about magnitude (e.g., $1/4 + 1/4 = 1/2$, so $1/2 > 1/4$) that may contribute to children's conceptual understanding of fractions (Bailey et al., 2015; Rittle-Johnson et al., 2001).

A mature understanding of fractions emerges as children integrate their understanding of fraction magnitudes with their understanding of whole number magnitudes (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014; Torbeyns, Schneider, Xin, & Siegler, 2015). Children with a strong grasp of whole numbers should then be better situated to make this integration than other children. Indeed, children who have a strong understanding of the magnitudes of whole numbers are more likely than other children to understand fractions as magnitudes, controlling for other numerical knowledge and general cognitive competencies, such as IQ (Bailey et al., 2012; Bailey et al., 2014; Hansen et al., 2015; Jordan et al., 2013; Siegler & Pyke, 2013; Vukovic et al., 2014).

1.2. Current study

Our overarching goal was to explore prior mathematics knowledge and cognitive factors that contribute to children's understanding of how fractions are represented on the mathematical number line and cross-grade gains in this understanding. Among these factors, we particularly tested the hypothesis that children's understanding of the relation between numerators and denominators, as well as whole numbers as magnitudes contribute to their understanding of fractions magnitudes in the context of the number line. We focused on the latter as the outcome of interest because the number line itself is an important foundational aspect of mathematics and integrating knowledge of fractions magnitudes with their understanding of the number line is a critical step in student's learning of more complex mathematics. The study differs from most previous studies with the focus on fractions knowledge just before or at the time students are taking high school algebra and with the exploration of factors contributing to the across-grade improvement in this knowledge. We also simultaneously considered the contributions of domain general abilities that predict competence with fractions or mathematics achievement generally; specifically, intelligence, the central executive component of working memory, and in-class attentive behavior (Deary, Strand, Smith, & Fernandes, 2007; Geary, 2011; Geary, Hoard, Nugent, & Bailey, 2013; Hansen et al., 2015; Jordan et al., 2013; Vukovic et al., 2014).

To identify the optimal set of predictors of 8th and 9th graders' performance on the fractions number line and improvements in performance across grade, we used traditional regression approaches and Bayes factors, a standard Bayesian model comparison method (Dienes, 2014; Gallistel, 2009; Jeffreys, 1961; Raftery, 1995; Rouder & Morey, 2011, 2012; Rouder, Speckman, Sun, Morey, & Iverson, 2009; Wagenmakers, 2007). The Bayes factor is the probability of the observed data under one model relative to another. With this approach we can compare one model comprised of IQ and in-class attention, for example, with another model comprised of IQ alone. The comparison provides an odds ratio between the two models. A ratio of 5, for instance, indicates that the data are 5 times more probable when in-class attention is included than when it is excluded.

In the current study, there were several advantages to including Bayes factors with regression models (Rouder & Morey, 2012; Wagenmakers, 2007). First, Bayes factors are evidence expressed as odds. When one model favors another with small odds (e.g., a ratio of 2), the condition may be reported without committing to one model or rejecting the other, and yet the quality of evidence is immediately interpretable. Although we do use rule-of-thumb procedures for discriminating Bayes models (below), we present all of the information needed for interested readers to evaluate the quality of fit of alternative models vis-a-vis their own priors regarding the importance of one potential predictor or another. Second, Bayes factors are interpretable even when there is collinearity between predictors. Specifically, the Bayes factor is higher for models with one of two highly correlated variables relative to models with both or none, providing interpretable (based on odds ratios) evidence for the relative importance of one variable or the other but not both. Moreover, by computing Bayes factors we can compare models containing every combination of different predictors available in the study and select the model comprised of the best set of predictors. As with any data set, this of course capitalizes on chance. To reduce false positives, we identified variables that emerged in all top models. These are variables that consistently predict fraction magnitude knowledge, regardless of which other predictors are included in the model.

2. Methods

2.1. Participants

The current study is part of a longitudinal assessment of developmental changes in children's mathematics knowledge from kindergarten to high school (see Geary, Hoard, Nugent, & Bailey, 2012). Two hundred and eighty-eight children were recruited from the public school system in Columbia, MO and finished the first year of testing. The 122 (49% males) 9th graders who completed all 14 tasks used in the current study were included in these analyses (the algebra test score of one child was missing and this data was replaced with the group mean of the remaining 121 children), and these students had higher intelligence scores ($M = 102$, $SD = 15$) in 1st grade than the children who did not complete all 14 tasks ($M = 96$, $SD = 15$), $t(286) = 3.52$, $p = 0.001$; Bayes factor = 43.7 in favor of difference between groups (see the rule-of-thumb below). These students, however, had similar first grade mathematics achievement ($M = 94$, $SD = 12$) scores to the students who were not included ($M = 91$, $SD = 13$), $t(286) = 1.57$, $p = 0.119$; Bayes factor = 2.4, in favor of no difference. The final sample averaged 6 years 10 months of age (range: 6y2 m–7y 11m) at the time of the first assessment in 1st grade, and 15 years 2 months (range: 14y7 m–16y 2m) at the time of the 9th grade fraction number line assessment.

2.2. Measures

2.2.1. General cognitive competencies

2.2.1.1. Intelligence. Verbal and nonverbal intelligence were assessed in 1st grade using the Vocabulary and Matrix Reasoning subtests of the *Wechsler Abbreviated Scale of Intelligence* (*WASI*; Wechsler, 1999), respectively. The scores were used to estimate IQ based on norms presented in the manual ($M = 100$, $SD = 15$).

2.2.1.2. Central executive. The central executive was assessed using three dual-task subtests of the *Working Memory Test Battery for Children* (WMTB-C; Pickering & Gathercole, 2001) in 1st and 5th grade. Listening Recall requires the child to determine if a sentence is true or false and then to recall the last word in each of a series of sentences (1-6 sentences). Counting Recall requires the child to count a set of 4, 5, 6, or 7 dots on a card and then to recall, in order, the number of dots counted on each card at the end of that series of cards (1-7 cards). Backward Digit Recall requires the child to recall a series of digits in the reversed order (2-7 digits). The subtests consist of span levels each containing a number of items to be remembered (see above), and each span level has six trials. Failing three trials at one span level terminates the subtest, and passing four trials moves the child to the next level. The total number of trials answered correctly was used as the central executive measure, because these scores are more reliable than span scores (α = 0.75 and 0.68 in 1st and 5th grade, respectively).

2.2.1.3. In-class attention. In-class attention was assessed with the Strength and Weaknesses of ADHD Symptoms and Normal Behavior (SWAN, Swanson et al., 2008), in which teachers rate the child's attention and hyperactivity relative to other children on a 1 (far below) to 7 (far above) scale for nine items. In-class attention was evaluated once a year from 2nd to 4th grade, inclusive. Given that some children did not have complete scores from 2nd to 4th grade, the average score across the three years (or across two or one year if data in other years were missing) was used (α = 0.88).

2.2.1.4. Mathematics achievement. Mathematics achievement was assessed in 7th grade with the Numerical Operations subtests of the *Wechsler Individual Achievement Test-II: Abbreviated* (WIAT-II: Wechsler, 2001). The Numerical Operations items include numeral writing, number discrimination, rote counting, and basic arithmetic operations. More difficult items include rational numbers and simple algebra and geometry problems.

2.2.1.5. Algebra achievement. To evaluate whether the fractions measures assess competencies that are foundational to algebra, as suggested by the NMAP (2008), we included a 9th grade algebra assessment. The assessment included 25 multiple-choice problems from Star et al.'s (2015) test that has been shown to be sensitive to individual differences in algebra learning (Rittle-Johnson & Star, 2009; Rittle-Johnson, Star, & Durkin, 2009). The items included standard solve-for-*x* problems, systems of equations, factoring, determining equation slope, and concept questions (e.g., definition of a nonvertical line). The sum of the item scores created a highly reliable composite (α = 0.87); the final score was the number correct minus the number incorrect.

2.2.2. Mathematical tasks

We measured 8th and 9th graders' understanding of how fractions magnitudes are situated on the number line using a standard procedure (Siegler et al., 2011). We also included 1st grade measures of whole number knowledge – the numbers sets test and number line task – that are predictive of later mathematics achievement (Bailey et al., 2014; Geary, 2011; Geary et al., 2013), and because in theory whole number knowledge should eventually be integrated with fraction magnitude knowledge (Siegler et al., 2011). The same measures in 6th grade were included because it is when many children enter middle school and because this is when our number line task changed from a 0–100 line to a 0–1000 line. It is possible that children who do not have a strong understanding of whole numbers would perform relatively poorly on this expanded number line and thus it might provide additional predictive information.

2.2.2.1. Number line estimation for fractions. Children in 8th and 9th grades received a computerized version of the fraction number line task. Each number line was 25 cm and had a start point of 0 and an endpoint of 5 with a target number centered 5.5 cm above it (72 pt. font). Following Siegler et al. (2011), target fraction stimuli were 1/19, 4/7, 7/5, 13/9, 8/3, 11/4, 10/3, 7/2, 17/4, and 9/2, drawing equally from each tenth of the number line. To ensure the child understood the task, two practice trials were administered. In the first trial, the child was presented with a number line with endpoints of 0 and 1 and with a target of ½, and was asked to click the mouse to indicate where ½ would go. The child's placement was then compared to a number line on the next screen with the ½ point correctly marked, and the experimenter discussed with the child how *"½ is half of one so we put it halfway between 0 and 1 on the number line."* In the second trial, a number line with endpoints of 0 and 5 with increments of 1 was presented, and the child was told, *"Here is a number line that goes from 0 to 5 and is marked in increments of 1."* An identical number line but without the incremental markings was next presented and the child was asked to place the target of ½ again. The child received feedback just as in the first practice trial, and was told how to change the response if desired. After the practice trials, the 10 test trials without feedback were then presented (α = 0.82 and 0.86 in 8th and 9th grade, respectively).

2.2.2.2. Number sets. This task was used to assess the speed and accuracy of representing and adding numerosities of sets of objects and Arabic numerals in 1st and 6th grades (Geary, Bailey, & Hoard, 2009; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). This task involves two types of number set stimuli: objects (e.g., stars and circles) in a half-inch square and an Arabic numeral (18pt font) in a half-inch square. These stimuli are combined into domino-like rectangles in the following combinations: objects/same objects, objects/different objects, objects/Arabic numerals, and Arabic numerals/Arabic numerals. The child's task is to circle any rectangles in which the stimuli sum to match a target number displayed on the top of a page. For practice, two items are presented to the child to choose from using a target number of 4. After the child's choice(s) has been made and discussed, eight more practice items are administered for the target number of 3. In the test, 36 items are presented on each of two pages for the target numbers of 5 and 9. On each page, there are 18 items matching the target and 18 items not matching. Six of these items contain 0 or an empty square. The child is asked to move across each line of the page from the left to right without skipping any, and to "circle any groups that can be put together to make the top number, five (or nine)" and to "work as fast as you can without making many mistakes."

Using a stopwatch, the child is given 60 and 90 s per page for the targets 5 and 9, respectively, and is asked to stop at the time limit. Because the children's performance was consistent across target number and item content (objects and Arabic numerals), the scores were combined to create an overall frequency of hits (α = 0.92 and 0.80 in 1st and 6th grade, respectively) and false alarms ($\alpha = 0.82$ and 0.46 in 1st and 6th grade, respectively) (Bailey et al., 2012; Geary et al., 2007, 2012). The scores were also adjusted for processing speed by using [hits – false alarms]/reaction time (Bailey et al., 2012). Very few of the younger children (e.g., 1st graders) were able to finish the task in the allotted time (Geary et al., 2007, 2009), and thus the maximum reaction time limit (300 s in total; 120 and 180 s for the target of 5 and 9, respectively) was used for each child. For older children (e.g., 6th graders) who typically completed the test within the time limit, each individual's actual reaction time was used.

2.2.2.3. Number line estimation for whole numbers. Children in 1st and 6th grades were asked to place target numbers on a number line. First graders were presented with a 25 cm number line with 0 and 100 on the two ends and a target number 5.5 cm above the middle of the line (72 pt. font). As a practice trial, the child first was asked to locate the target number of 50 on the line and received feedback from the experimenter (e.g., "[Great/Actually,] *the number 50 is half of 100, so we put it halfway in between 0 and 100 on the number line*"). Following Siegler and Booth (2004), the child then was tested with 24 target numbers without feedback (3, 4, 6, 8, 12, 17, 21, 23, 25, 29, 33, 39, 43, 48, 52, 57, 61, 64, 72, 79, 81, 84, 90, and 96). The 6th graders received a 0–1000 number line task with 24 test numbers (33, 48, 61, 84, 126, 170, 215, 247, 253, 298, 336, 391, 425, 480, 524, 577, 619, 642, 721, 798, 816, 843, 905, and 964). A pencil-and-paper version of the task was used in first grade and a computerized version, where the children moved the cursor to locate the target numbers, was used thereafter. Children's accuracy of number line estimation was determined by calculating their mean percent absolute error (PAE = |Estimate – Target Number|/Range of Estimates; Siegler and Booth, 2004) ($\alpha = 0.75$ and 0.77 in 1st and 6th grade, respectively).

2.2.2.4. Fraction conceptual knowledge of numerator-denominator relations test. Children's conceptual understanding of fractions was assessed in 7th grade with a 12-item fraction comparison task, in which the child circled the fraction with a larger value from two fractions in 120 s (Geary et al., 2013). This task consists of three types of comparisons, each with 4 items.[1] The first type presents two fractions with a constant numerator but different denominators (e.g., 1/5 vs. 1/9), which assessed the child's understanding of the inverse relation between the denominator and the fraction value. In the second type of comparison, numerators and denominators are reversed (e.g., 3/2 vs. 2/3) to assess whether the child understands that the larger fraction should have the larger numerators and the smaller denominators. The comparisons in the third type involve a fraction with a 1/2 value as an anchor and the other fraction close to 1 (e.g., 20/40 vs. 8/9). A skilled child is expected to quickly identify the 1/2 fraction and choose the other one.

The three types of comparisons were designed to examine whether the children conceptually understand the meanings of the numerator, the denominator, and the value of a fraction as a whole. The child received 1 point for circling the correct answer in each pair. Because scores were significantly correlated across the three comparison types ($rs > 0.41$, $ps < 0.001$), they were summed to form a single score to represent children's conceptual understanding of the numerator-denominator relation ($\alpha = 0.66$).

2.2.2.5. Fraction procedural knowledge test. Based on Hecht (1998), the children were administered three fraction arithmetic tests in 7th grade: Addition (e.g., 2 ¼ + 1/4), Multiplication (e.g., $1/4 \times 1/8$), and Division (e.g., $1/3 \div 1/6$). For each test, the score was the number of problems solved correctly in 1 min. However, performance on the Multiplication and Division tests was highly skewed and poor (e.g., the median correct for Division was 0). Thus, we used number correct for the 12 fraction addition problems.

2.3. Analysis

We computed Bayes factors for regression models following Liang, Paulo, Molina, Clyde, and Berger (2008) and Rouder and Morey (2012). The models were implemented using Morey and Rouder's (2014) BayesFactor package (version 0.9.11-1) for R. We note alternative models for 8th grade as $M8_m$, where m = the specific set of predictors, and comparisons as $B8_{mn}$, where B represents Bayes factor (i.e., ratio of alternative models), and m and n are alternative models. Returning to our earlier example, $M8_1$ might be comprised of IQ and in-class attention and $M8_2$, comprised of IQ alone. The comparison provides an odds ratio between the two models and is symbolized as $B8_{12}$; the order of the subscripts denotes the numerator and denominator, respectively, for calculating the ratio. The model $M8_0$ represents the null, that is, a model in which none of the evaluated predictors are probable. We used the same notations for 9th grade and the cross-grade improvement, but substituted 9 and I for 8, respectively, for these models.

Jeffreys (1961) provides a rule-of-thumb where Bayes factor evidence values of 1/3 (0.33) are considered substantive and values of 1/10 (0.10) or less are considered strong. In our analyses, we first identified the best set of predictors and

[1] There was one more type of comparison in this test, in which both numerators and denominators differ and the fraction with the larger value always has the larger numerator and smaller denominator (e.g., 3/10 vs. 2/12). The ratio between the numerators in each comparison is 1.5 and the ratios between the denominators range from 1.1 to 1.25; these ratios were chosen based on the Weber function for magnitude discrimination in adolescents (Halberda and Feigenson, 2008). We excluded these items because the correlation between them and the three other item types was lower than the correlations among the three latter types. We included the fourth type of item in follow-up data analyses and this did not change the main findings of this study.

Table 1
Descriptive statistics and grade of task administration.

	Grade 1	2–4	5	6	7	8	9
IQ1	102 (15)						
Attn3		5 (1)					
CE	34 (9)		51 (10)				
Math7					97 (19)		
WNL1	14 (7)						
WNL6				4 (2)			
Numset	0.11(.04)			0.28 (.06)			
Frac_c7					11 (2)		
Frac_a7					7 (4)		
FNL						14 (10)	12 (10)
ALG9							15 (6)

Note: Numbers at the end of the variable names represent the grade in which the data were obtained. Numbers in the parentheses indicate the standard deviations. IQ = standardized intelligence score; Attn = in-class attention on a 1–9 scale; CE = central executive (total correct); Math = standardized mathematics achievement; WNL = whole number number line (percent error); Numset = number sets; Frac_c = fraction concepts (maximum score = 12); Frac_a = fraction addition (maximum score = 12); FNL = fraction number line (percent error); ALG = algebra achievement (maximum score = 25).

then compared this model to a model in which one of the predictors was dropped or an alternative predictor was added. The odds ratio then identifies how much worse the fit becomes as specific predictors are excluded from or included with the best model. For instance, if the Bayes factor is 1/10 as large in the model with the dropped predictor, then the data are 10 times as probable under the favored model than under the model without the predictor. As noted, we also include standard regression techniques. The combination provided a more complete evaluation of the utility of potential predictors than either approach could provide alone.

2.4. Procedure

Most children were assessed individually in a quiet location at the school site for all tasks except the fractions procedural knowledge (7th grade) and algebra (9th grade) tests were part of group-administered assessments (Geary et al., 2013; Geary, Hoard, Nugent, & Rouder, 2015). The WMTB-C was administered in a mobile testing van that parked outside the child's house after school hours or on weekends. The grades for task administration and mean scores are shown in Table 1.

3. Results

3.1. Accuracy of fraction number line estimation and the prediction of algebra achievement

The accuracy of 8th and 9th graders' fraction magnitude estimation was highly correlated across grades ($r(120) = 0.79$, $p < 0.001$) and substantially improved from 8th to 9th grade (PAE = 14% and 12%, respectively; $t(121) = 3.22$, $p = 0.002$; Bayes factor = 13.1 in favor of difference between grades). The 8th graders' estimates for each target fraction are shown in Fig. 1 (9th graders' response patterns were very similar), and a repeated measures ANOVA with the 10 target fractions as the within-subject variable showed that accuracy differed across the targets ($F(5.7, 690.3) = 9.18$, MSE = 0.028, $p < 0.001$); specifically, accuracy was higher near the endpoints and lower in the center.

To assess the predictive validity of our outcome measures, we assessed whether 8th and 9th grade accuracy for the fraction number line task was correlated with 9th grade algebra achievement, controlling for sex, age in 8th grade, IQ in 1st grade, central executive scores in 1st and 5th grade, and in-class attention in 2nd-4th grade. The partial correlations were −0.371 ($p < 0.001$) and −0.346 ($p < 0.001$) for 8th and 9th grade fraction number line accuracy, respectively, indicating they were highly predictive of algebra achievement; we do not present Bayes factors for partial correlations due to computational difficulties. We also examined the partial correlation between algebra achievement and the fraction number line measure in one grade, with the fraction number line measure in the other grade included as an additional covariate. The partial correlation was strong for 8th grade ($pr = -0.213$, $p = 0.022$) but weaker for 9th grade ($pr = -0.159$, $p = 0.090$).

Including 7th grade mathematics achievement as one more covariate (i.e., general mathematics achievement prior to administration of the 8th grade fractions measure), the significant correlations remained for the both 8th ($pr = -0.241$, $p = 0.009$) and 9th ($pr = -0.235$, $p = 0.012$) grade fraction number line measures when the fraction measure in the other grade was not included as a covariate. The zero-order correlation between 7th grade mathematics achievement and 9th grade algebra scores was high ($r = 0.78$, $p < 0.001$). However, neither partial correlation was significant when both fraction number line measures were entered as predictors ($pr = -0.132$, $p = 0.161$ for 8th grade and $pr = -0.119$, $p = 0.207$ for 9th grade), due to the high correlations among the fraction number line and mathematics achievement measures ($|r|s > 0.61$, $ps < 0.001$; Table 2). These results suggest that students' understanding of the position of fractions magnitudes on the number line is uniquely related to later algebra learning, even when earlier general mathematics achievement and domain-general cognitive abilities are controlled.

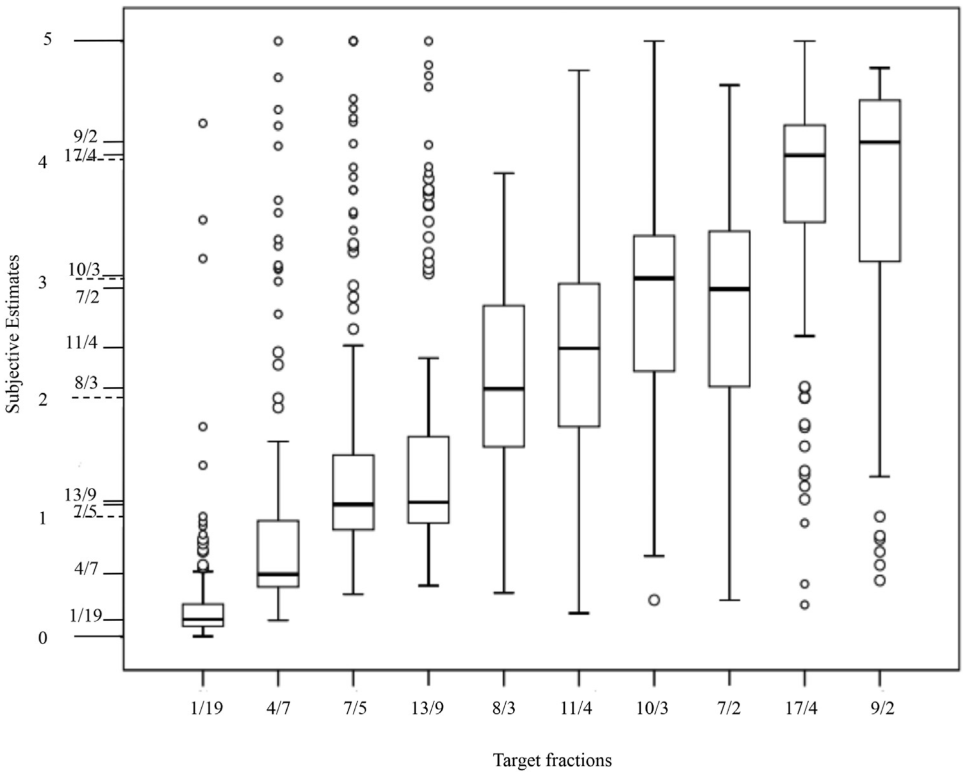

Fig. 1. Boxplots for the 8th graders' estimates for the 10 target fractions. The median estimate and the corresponding position for each target fraction is presented on the Y-axis.

Table 2
Correlations among variables.

	Sex	Age8	IQ1	Attn3	CE1	CE5	Math7	WNL1	WNL6	Frac_a7	Frac_c7	Numset1	Numset6	FNL8
Age8	0.123													
IQ1	0.234	−0.146												
Attn3	−0.233	0.069	0.218											
CE1	0.034	0.055	0.388	0.565										
CE5	−0.002	−0.095	0.449	0.570	0.599									
Math7	0.122	−0.087	0.525	0.508	0.491	0.536								
WNL1	−0.281	−0.196	−0.430	−0.320	−0.440	−0.414	−0.447							
WNL6	−0.067	0.087	−0.216	−0.334	−0.223	−0.322	−0.487	0.283						
Frac_a7	0.172	−0.038	0.365	0.357	0.237	0.349	0.586	−0.415	−0.290					
Frac_c7	0.113	0.030	0.425	0.407	0.355	0.426	0.444	−0.496	−0.267	0.355				
Numset1	0.222	0.076	0.482	0.390	0.465	0.431	0.580	−0.593	−0.272	0.542	0.445			
Numset6	0.069	0.020	0.302	0.537	0.403	0.432	0.661	−0.462	−0.359	0.560	0.400	0.731		
FNL8	−0.207	0.013	−0.530	−0.423	−0.431	−0.510	−0.623	0.518	0.291	−0.512	−0.571	−0.521	−0.472	
FNL9	−0.188	0.052	−0.514	−0.454	−0.458	−0.525	−0.611	0.537	0.474	−0.505	−0.561	−0.553	−0.480	0.787

Note: Numbers at the end of the variable names represent the grade in which the data were obtained. IQ = standardized intelligence score; Attn = in-class attention; CE = central executive; Math = standardized mathematics achievement; WNL = whole number number line; Frac_c = fraction concepts; Frac_a = fraction addition; Numset = number sets; FNL = fraction number line.

3.2. Predictors of fraction number line accuracy

As shown in Table 2, children's accuracy in placing fractions on the number line was correlated with all potential predictors measured before 8th grade ($|r|$s > 0.18, p < 0.039), except for age, and all of the potential predictors were intercorrelated, except for age and sex. In this situation, Bayes factors provide a useful adjunct to standard procedures in the identification of the best set of correlated predictors. We compared the 8192 models formed by all possible combination of the 13 predictors. Table 3 shows the best models to emerge from this analysis. The values of $B8_{m0}$ are large for all models, providing strong evidence against the null. The last column shows $B8_{m1}$, the Bayes factor between a particular model, $M8_m$, and the best fitting model, denoted $M8_1$. As noted, the value of $B8_{m1}$ indicates the probability of the data under $M8_m$ relative to that under $M8_1$.

Table 3
Bayes factor analysis of 8th grade fraction magnitude knowledge.

Model	$B8_{m0}$	Excluded	$B8_{m1}$
$M8_1$ Frac_c7 + Math7 + IQ1 + Frac_a7	9.56×10^{15}	–	1
$M8_2$ Math7 + IQ1 + Frac_a7	2.34×10^{13}	Frac_c7	0.002
$M8_3$ Frac_c7 + IQ1 + Fract_a7	3.07×10^{14}	Math7	0.032
$M8_4$ Frac_c7 + Math7 + Frac_a7	3.66×10^{15}	IQ	0.383
$M8_5$ Frac_c7 + Math7 + IQ1	7.45×10^{15}	Frac_a7	0.779
$M8_6$ Frac_c7 + Math7	2.70×10^{15}	Frac_a7 + IQ1	0.282
$M8_7$ Frac_c7 + Math7 + IQ1 + WNL1	8.99×10^{15}		0.940
$M8_8$ Frac_c7 + Math7 + IQ1 + Frac_a7 + WNL1	7.15×10^{15}		0.748

Note: Frac_c7 = fraction concept in 7th grade; Frac_a7 = fraction addition in 7th grade; Math7 = standardized mathematics achievement in 7th grade; IQ1 = intelligence in 1st grade; WNL1 = Whole number line in 1st grade. The variables in the Excluded column indicate the variable excluded from $M8_1$.

Model $M8_1$, – comprised of fraction concepts, fraction addition and mathematics achievement in 7th grade, and IQ in 1st grade – was the best fitting model among all alternatives. This model was then compared to more parsimonious models that excluded in turn each of these predictors. Models $M8_2$ to $M8_5$, were identical to $M8_1$ except for a single excluded variable. Excluding fraction concepts in 7th grade, for example, resulted in $M8_2$ and the Bayes factor $B8_{21}$ of 0.002 indicates the data were 0.2% as probable under the model with fraction concepts excluded. In addition to fraction concepts, the Bayes factors provided very strong evidence for the inclusion of mathematics achievement in 7th grade ($M8_3$; $B8_{31} = 0.032$). Excluding IQ in 1st grade ($M8_4$) or fraction addition in 7th grade ($M8_5$) resulted in the data 38.3-77.9% as probable relative to the model including it ($B8_{41} = 0.383$ and $B8_{51} = 0.779$), providing only weak evidence in favor of retaining these predictors. However, excluding both IQ and fraction addition ($M8_6$) resulted in the data being less than 1/3 times as probable relative to the model including both ($B8_{61} = 0.282$), suggesting one of the two predictors should be retained. Given the exclusion of IQ influenced the probability of observed data to a larger extent than the exclusion of fraction addition, IQ was retained ($M8_5$).

We also compared Model $M8_1$ with the second-best 4-predictor model, $M8_7$. This model included number line accuracy in 1st grade and excluded fraction addition in 7th grade. The resulting value, $B8_{71}$, was 0.940 indicating the data were 94.0% as probable under the second-best 4-predictor model as under $M8_1$. In addition, the data were 82.9% as probable under $M8_5$ relative to $M8_7$. We also considered all 5-predictor models formed by adding one more variable to $M8_1$. The best of these was Model $M8_8$, in which number line accuracy in 1st grade was added in Model $M8_1$. The resulting Bayes factor was 0.748 relative to $M8_1$ and 0.960 relative to $M8_5$. The gist is the data were less probable when more variables were included in the model, but that the evidence for excluding the number line accuracy in 1st grade was not strong.

In summary, $M8_5$ and $M8_1$ may be retained as describing the combination of variables that were best correlated with 8th graders' accuracy in placing fractions on a number line. Using Jeffreys' (1961) rule-of-thumb, the models with and without fraction addition in 7th grade are not clearly distinguishable ($M8_1$ and $M8_5$). In addition, the models that replaced this variable with the number line accuracy for whole numbers in 1st grade ($M8_7$) or involved both fraction addition in 7th grade and the number line accuracy in 1st grade ($M8_8$) were not substantively different from the model that excluded them ($M8_5$). For parsimony, we concluded $M8_5$, represented the best set of predictors of the models shown in Table 3.

To reduce false positives given the large number of models computed in these analyses, as a follow up we also examined the top fourteen models (not all are shown in Table 3) that resulted in Bayes factors that were greater than 1/3 that of Model $M8_1$. Fractions concepts and mathematics achievement in 7th grade were included in all of them, indicating that these variables always emerged in best fitting models regardless of which other covariates were also included in the models. Intelligence emerged in eight of these alternative models, indicating its inclusion or not depends on the other variables in the models. Table 4 shows the results of a multiple regression analysis with all of the 13 variables ($R^2 = 0.568$). As shown, fraction concepts and mathematics achievement in 7th grade emerged as significant effects, confirming the Bayes analyses. In these analyses, however, IQ was not significant using conventional p values ($p = 0.148$), suggesting the importance of IQ may be underestimated using traditional methods and correlated predictors.

We followed the same procedure for predicting 9th graders' number line accuracy. As shown in Table 5, the best fitting model ($M9_1$) included fraction concepts, fraction addition and IQ that emerged for 8th grade accuracy. Mathematics achievement in 7th grade was no longer included in this model, but the number line accuracy for whole numbers in 6th grade and the central executive score in 1st grade were included. Models $M9_2$ to $M9_6$ examined the effect of individually excluding each of the five variables in $M9_1$. The resulting values of Bayes factors varied from 0.006 to 0.252, indicating that the data were much less probable under the models excluding any one of the five predictors than the model including all of them, and thus all variables were retained. In addition, comparing $M9_1$ with the second-best 5-predictor model (Model $M9_7$) suggested that $M9_7$ was not more predictive than $M9_1$ ($B9_{71} = 0.571$). Finally, we compared $M9_1$ with $M9_8$, the best 6-predictor model formed by adding number line accuracy for whole numbers in 1st grade in $M9_1$. This resulted in the Bayes factor of 0.651, indicating that data were not more probable when more predictors were added in the model. In short, $M9_1$ was the best fitting model for the accuracy of 9th graders' fraction number line estimation.

Again, to reduce false positives we examined the eighteen models (not all are shown in Table 5) with Bayes factors greater than 1/3 of $M9_1$. Fractions concepts in 7th grade and number line accuracy for whole numbers in 6th grade were included in all of them, indicating the importance of these variables regardless of which other covariates are included in

Table 4
Multiple regression models predicting 8th and 9th grade fraction number line accuracy and cross-grade improvement.

	8th grade		9th grade		Improvement	
	Estimates	ΔR^2	Estimates	ΔR^2	Estimates	ΔR^2
Frac_c7	−0.239 (.08)**	0.036**	−0.202 (.08)**	0.026**	0.045 (.11)	0.001
Frac_a7	−0.136 (.09)	0.010	−0.126 (.08)	0.009	0.009 (.12)	0.000
Math7	−0.261 (.11)*	0.023*	−0.111 (.10)	0.004	0.218 (.15)	0.016
IQ1	−0.126 (.09)	0.009	−0.102 (.08)	0.006	0.030 (.12)	0.000
WNL1	0.104 (.09)	0.005	0.109 (.09)	0.006	0.014 (.13)	0.000
WNL6	−0.050 (.07)	0.002	0.200 (.07)**	0.029**	0.386 (.11)***	0.108***
CE1	−0.005 (.09)	0.000	−0.048 (.09)	0.001	−0.068 (.13)	0.002
CE5	−0.100 (.09)	0.005	−0.079 (.09)	0.003	0.027 (.13)	0.000
Numset1	−0.021 (.11)	0.000	−0.131 (.11)	0.006	−0.173 (.16)	0.010
Numset6	0.050 (.11)	0.001	0.103 (.11)	0.003	0.085 (.16)	0.002
Attn3	−0.080 (.10)	0.003	−0.097 (.10)	0.004	−0.030 (.14)	0.000
Age8	0.007 (.07)	0.000	0.050 (.07)	0.002	0.068 (.10)	0.004
Sex	−0.087 (.07)	0.006	−0.067 (.07)	0.003	0.027 (.10)	0.001

Note: Numbers are the standardized parameter estimates from the multiple regression models, and numbers in the parentheses are standard errors. * $p < 0.05$. ** $p < 0.01$. *** $p < 0.001$. ΔR^2 = change of the coefficient of determination when all other variables entered in the regression model. Numbers at the end of the variable names represent the grade in which the data were obtained. Frac_c = fraction concepts; Frac_a = fraction addition; Math = standardized mathematics achievement; IQ = standardized intelligence score; WNL = whole number number line; CE = central executive; Numset = number sets; Attn = in-class attention. Because children's fraction number line accuracy was measured with the percent absolute error, negative parameter estimates mean that higher scores on the independent variable predict smaller errors.

Table 5
Bayes factor analysis of 9th grade fraction magnitude knowledge.

Model	$B9_{m0}$	Excluded	$B9_{m1}$
$M9_1$ Frac_c7 + Frac_a7 + IQ1 + WNL6 + CE1	4.24×10^{16}	–	1
$M9_2$ Frac_a7 + IQ1 + WNL6 + CE1	4.26×10^{14}	Frac_c7	0.010
$M9_3$ Frac_c7 + IQ1 + WNL6 + CE1	2.08×10^{15}	Frac_a7	0.049
$M9_4$ Frac_c7 + Frac_a7 + WNL6 + CE1	8.53×10^{15}	IQ1	0.201
$M9_5$ Frac_c7 + Frac_a7 + IQ1 + CE1	2.44×10^{14}	WNL6	0.006
$M9_6$ Frac_c7 + Frac_a7 + IQ1 + WNL6	1.07×10^{16}	CE1	0.252
$M9_7$ Frac_c7 + Frac_a7 + IQ1 + WNL6 + CE5	2.42×10^{16}		0.571
$M9_8$ Frac_c7 + Frac_a7 + IQ1 + WNL1 + WNL6 + CE1	2.76×10^{16}		0.651

Note: Frac_a7 = fraction addition in 7th grade; Frac_c7 = fraction concept in 7th grade; IQ1 = intelligence in 1st grade; WNL6 = Whole number line in 6th grade; WNL1 = Whole number line in 1st grade; CE1 = executive functions in 1st grade; CE5 = executive functions in 5th grade; Numset1 = number sets in 1st grade. The variables in the Excluded column indicate the variable dropped from $M9_1$.

Table 6
Bayes factor analysis of the improvement of fraction magnitude knowledge from 8th to 9th grade.

Model	BI_{m0}	BI_{m1}
MI_1 WNL6	35.26	1
MI_2 Numset1	0.27	0.008
MI_3 WNL6 + Math7	31.63	0.897

Note: WNL6 = Whole number line in 6th grade; Numset1 = number sets in 1st grade; Math7 = standardized mathematics achievement in 7th grade.

the model. Intelligence was included in fifteen of these models, fraction addition in twelve of them, and the 1st grade central executive in four of them. A regression analysis with all 13 variables confirmed the importance of 7th grade fractions concepts and number line accuracy for whole numbers in 6th grade ($R^2 = 0.597$; Table 4). Notably, other variables including IQ, fraction addition and central executive that showed important predictive effects in the Bayes analysis did not emerge in the regression analysis, likely due to collinearity. The identification of IQ in most of the top Bayes models suggests this is an important variable, despite the lack of conventional significance in the regression. We should also consider that fraction addition may be important based on the Bayes models, but with less consistent evidence for the central executive.

Finally, we sought to identify variables that predicted 8th to 9th grade improvement in the accuracy of fraction number line placements. The improvement score was the difference of the child's PAE between the 9th and the 8th grade. The Bayes factor analysis presented in Table 6 shows that the best fitting model (MI_1) included whole number line accuracy in 6th grade. Data were less probable under the second-best 1-predictor model including only number sets in 1st grade, denoted MI_2 ($BI_{21} = 0.008$). The best 2-predictor model, Model MI_3, was formed by adding mathematics achievement in 7th grade into MI_1, and the resulting Bayes factor was 0.897, indicating the data were slightly less probable when mathematics achievement was included but the evidence for excluding this variable was not strong.

We then examined the three other models with Bayes factors greater than 1/3 of MI_1, and the number line accuracy for whole numbers in 6th grade was included in all of them. The associated regression for the cross-grade improvement confirms the importance of whole numbers in 6th grade ($R^2 = 0.132$; Table 4)

4. Discussion

The National Mathematics Advisory Panel concluded "by the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line" (NMAP, 2008, p. 20). Contra this goal and consistent with previous research (Bailey et al., 2014; Siegler et al., 2011), we found that some 8th and 9th graders still had serious difficulties representing the magnitudes of fractions on number lines, despite high accuracy in the placement of whole numbers on the line. Children's difficulties in this study and others indicate that many of them have not yet achieved the conceptual insight that fractions, as with whole numbers represent quantitative magnitudes that can be situated on the number line (Siegler et al., 2011; Siegler & Pyke, 2013). Consistent with conclusions of the NMAP (2008), our results confirm that this poor understanding of fraction magnitudes will compromise the learning of algebra in 9th grade. The latter highlights the importance of exploring the constellation of traits and knowledge that may be the foundations for children's integration of their knowledge of fraction magnitudes with their knowledge of the number line. In the present study, we provided data to comprehensively identify sets of numerical and domain-general factors important for students' understanding of how fraction magnitudes are situated on the number line and cross-grade gains in this understanding.

4.1. Mathematical foundations

Seventh graders' conceptual understanding of the relation between the numerator and denominator emerged as a critical predictor of their ability to situate fractions on the number line in 8th and 9th grades, in keeping with the finding that a conceptual understanding of fractions contributes to gains in fraction learning for most elementary and middle school students (Hallett et al., 2010; Hecht, 1998; Hecht, Close, & Santisi, 2003; Mazzocco & Devlin, 2008). Specifically, our fraction comparison task examined different aspects of conceptual understanding of fractions, including how the magnitude of a fraction changes when the numerator is given and the denominator varies (e.g., 1/5 vs. 1/9) or both components vary (e.g., 2/3 vs. 3/2), and the identification and understanding of fraction magnitudes close to ½ and 1 (e.g., 20/40 vs. 8/9). As mentioned earlier, comparing fractions may not necessarily involve representing the magnitude of each of the fractions in pair (though children may spontaneously represent fractions magnitudes), as long as children conceptually understand why and how a fraction is determined by the numerator-denominator relation (e.g., when the numerator is given, a fraction becomes numerically larger when the denominator becomes smaller). In short, performance on this comparison measure requires children to recognize that a fraction is determined by the numerator-denominator relation, rather than the independent whole numbers in the components; that is, they must inhibit the whole number bias that interferes with the learning of fraction magnitudes (Ni & Zhou, 2005; Van Dooren et al., 2015).

Notably, in the present study, children's conceptual understanding of numerator-denominator relations was measured in a timed task. The timed nature of the test required children not simply to be able to explicitly understand the meaning of the numerator and denominator but also fluently apply this relation when processing fractions. Seventh graders who quickly and accurately determined how fractions systematically varied in the relations between the numerator and denominator were the most successful at later placing fractions on a number line. The implication is that practice in applying conceptual knowledge to the domain of learning may be an important aspect of mastery, even if the initial conceptual insight itself often does not require procedure-like practice (Cooper & Sweller, 1987). Notably, although children's conceptual understanding of numerator-denominator relations is an important predictor of their later ability to place fractions on the number line, the former appears to be a necessary but not a sufficient component for the latter, given that several other numerical and non-numerical factors also contribute to fraction number line performance. Therefore, in addition to pointing out the importance of the understanding of numerator-denominator relations, the effects of other factors should also be emphasized.

Seventh graders' fraction arithmetic – their speed and accuracy at solving fractions addition problems – may also be important for coming to understand fractions as magnitudes and their positions on the number line, but the evidence for this was weaker than that for fractions concepts. In other words, our results suggest that measures of fraction arithmetic should be included in future longitudinal and experimental studies of children's learning of how fractions are situated on the number line but we cannot make a strong argument for its importance based on our results. In addition to our results, one reason for including fraction addition and fraction arithmetic more broadly is that transforming fractions may give children opportunities to compare the magnitudes of fractions involved in the operations. For instance, comparing 1/8 and 1/4 in the addition of 1/8 + 1/8 = 1/4 may strengthen children's understanding that the fraction with the smaller denominator actually has the larger magnitude. These outcomes may be important for overcoming the whole number bias during fractions learning (Ni & Zhou, 2005; Van Hoof et al., 2015). In any case, our results are consistent with several other studies showing competence in using fractions procedures is related to conceptual knowledge (e.g., Bailey et al., 2015; Rittle-Johnson et al., 2001), but conceptual knowledge is more proximal to children's developing competence with fractions (Hecht, 1998; Rittle-Johnson & Alibali, 1999; Siegler et al., 2011; Siegler and Lortie-Forgues, 2014), although we did not examine individual differences in this relation and thus it is possible that some children understand fractions concepts without competence in using associated procedures (Hallett et al., 2010).

The finding that 6th graders' accuracy in placing whole numbers on a number line predicted 9th graders' fraction number line performance is consistent with Siegler et al.'s (2011) model; specifically, that children's fraction and whole number development is integrated within their understanding of the quantitative magnitudes of these numbers (Bailey et al., 2014; Jordan et al., 2013; Siegler et al., 2011). However, competence with whole numbers was not strongly predictive of competence with fraction number line performance in 8th grade, although we could not rule out a contribution of 1st grade whole number line performance, based on the Bayes factors (see also Bailey et al., 2014). In any case, it was the only predictor of cross-grade gains of fraction magnitude knowledge, suggesting that integration of the representations of whole number and fraction magnitudes with respect to the number line occurs gradually and perhaps after children have some understanding of fraction magnitudes and can inhibit whole number biases (Van Hoof et al., 2015). These results are evidence for the importance of early instruction on fraction magnitudes, a competence that needs to be particularly emphasized in mathematics instruction in U.S. schools (Fuchs et al., 2013).

Alternatively, the correlation between whole number and fraction number line performance revealed in our study may be due to using the same task in both cases. However, the fact that the accuracy of placing whole numbers on the number line correlates with fraction conceptual understanding shown in the present study and with general fraction competencies shown in other studies (Bailey et al., 2014; Jordan et al., 2013; Siegler et al., 2011; Vukovic et al., 2014) suggest that whole number magnitude knowledge indeed plays an important role in fraction learning in general. Thus, the use of the number line task to measure whole number and fraction magnitude knowledge may not completely account for their correlation. Of course, studies that assess the relation between whole number knowledge and children's understanding of fraction magnitudes using tasks that do not involve the number line (e.g., ordering fractions) are needed to fully evaluate our suggestion.

One aspect of our results was inconsistent with Vukovic et al.'s (2014) finding that 1st graders' number sets performance predicted their conceptual understanding of fractions including magnitude knowledge in 4th grade. In our study, children's number sets performance, as assessed in 1st and 6th grades did not emerge as a strong predictor for their fraction number line performance in 8th or 9th grade. One potential reason is that we used a measure of conceptual understanding of fractions as a predictor (Siegler et al., 2011), and Vukovic et al. (2014) used a measure similar to our fractions concept test as the outcome. This difference in outcome measures may have obscured the effect of number sets performance in our study. Another but related potential reason may be that exact representations of small whole numbers play an important role in children's early conceptual understanding of fractions, as tested in Vukovic et al., but later this influence declines as children's conceptual competence improves. The same reason may explain why in-class attention showed salient predictive effects on 4th-5th graders' fraction conceptual understanding including magnitude knowledge (Hecht and Vagi, 2010; Hecht et al., 2003) but was not predictive for 8th and 9th graders' fraction magnitude knowledge in the present study. Unfortunately, we could not test this hypothesis because elementary-school measures of children's fraction knowledge were not included in the longitudinal study.

4.2. Domain-general competencies

The domain-general cognitive competencies – intelligence and the central executive component of working memory – emerged as predictors of fraction magnitude performance in one or the both grades, consistent with other recent studies (Hansen et al., 2015; Jordan et al., 2013). The predictive effects of IQ and the central executive for fractions number line performance of 9th graders were not revealed in traditional regression analyses with significance testing, but we indeed observed their important contributions to fractions number line performance with the Bayes factors. The implication is that significance testing may underestimate the contributions of the associated competencies, possibly due to the effects of collinearity on regression estimates.

These domain-general cognitive competencies may play different roles in the acquisition and improvement of students' understanding of how fraction magnitudes are situated on the number line. The finding that 1st graders' IQ predicted their fraction number line performance 7 and 8 years later is consistent with the well-established relation between intelligence and academic achievement and with previous longitudinal studies of the relation between IQ and mathematics learning (Deary et al., 2007; Geary, 2011; Jordan et al., 2013; Seethaler, Fuchs, Star, & Bryant, 2011). Intelligence however did not predict short-term, 8th to 9th grade gains in fraction number line performance. One potential reason is that intelligence may capture individual differences in students' understanding of fractions, but it may not capture within-student refinements of this knowledge. Another very likely reason is there is little variance to be explained by intelligence, as well as by other factors, in the cross-grade gains in fractions magnitude knowledge. To distinguish these possibilities and further explore factors contributing to cross-grade gains of fractions magnitude knowledge, the gains need to be measured in a wider range of grades in the future studies.

Our findings add to the results from previous studies showing that the central executive contributes to children's fraction knowledge (Bailey et al., 2014; Hansen et al., 2015; Hecht & Vagi, 2003; Jordan et al., 2013; Vukovic et al., 2014). In our results, the central executive was predictive for 9th but not 8th graders' fraction number line performance. It is possible that attentional control and focus contributes to children's ability to detect subtle differences between the magnitudes of similar fractions or refine the placement of fractions on number lines. We cannot however discriminate between these alternatives with the available data.

4.3. Limitations and implications

Our study included domain-general and domain-specific predictors of student's understanding of the number line but did not include all potential predictors (Bailey et al., 2014; Jordan et al., 2013; Vukovic et al., 2014) and focused on a critical but narrow age range (i.e., 8th–9th grade). As we noted with the comparison of our findings to those of Vukovic et al., studies using different predictors and age groups may yield different outcomes. Our fractions comparison test for instance only captures one aspect of children's conceptual knowledge and other measures (e.g., fractions are continuous rather than discrete like whole numbers; Van Dooren et al., 2015; Van Hoof et al., 2015) may yield different results. In addition, because of the small number of items on our fractions number line task, we could not examine magnitude knowledge for improper and proper fractions separately nor could we determine exactly how children were making the number line placements (e.g., using proportional reasoning; Cohen and Sarnecka, 2014; Rouder & Geary, 2014). Despite these limitations, the conjoint Bayes factors and traditional regression analyses enabled a more comprehensive analysis of the precursors of children's fraction magnitude knowledge than previous studies, and demonstrated that different competencies may be important at different points in children's skill development. Our findings also inform mathematics education by identifying instructional targets. Fluency in processing the relations between numerators and denominators, not just conceptually understanding the relation, emerged as a salient target. Finally, our results suggest that poor understanding of fraction magnitudes, as assessed by number line placements, in 8th grade may be diagnostic of later difficulties with high school algebra.

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